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ON CYCLES THROUGH PRESCRIBED VERTICES IN WEAKLY SEPARABLE GRAPHS

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1. Notions and notations

1.1. Given an undirected graph G let $V(G)$, $E(G)$, $\kappa(G)$ and comp(G) denote the vertex-set, edge-set, vertex connectivity and number of components of G, respectively. Put $P(G)=\{(X, Y)\colon X\subseteq V(G),\ T\subseteq E(G-X)\}$. For $X\subseteq V(G)$ let $G(X)$ denote the induced subgraph. For $Y\subseteq E(G)$ let $G(Y)$ denote the subgraph of G with the edge-set Y and without isolated vertices.

Given $T\subseteq V(G)$ a cycle C of G is called a *T-cycle* iff $T\subseteq V(G)$.

1.2. For $Y\subseteq E(G)$ let ∂Y denote the set of vertices covered by both Y and $E(G)-Y$, and put $c_G(Y)=\sum_s \lfloor \frac{1}{2}|\partial_G Y_s|\rfloor$ with the summation over all components of $G(Y)$, Y_s being the edge-set of the sth component. For $(X, Y)\in P(G)$ put $c_G(X, Y)=|X|+c_{G-X}(Y)$.

Definition [4]. Given $T\subseteq V(G)$, a pair (X, Y) from $P(G-T)$ is called a *T-separator* if T meets more than $c_G(X, Y)$ distinct components of $G-X-Y$.

2. Previous results

2.1. Clearly G cannot have both a T-cycle and a T-separator.

2.2. In [4] the following 'cycle-separator' alternative is established.

Theorem. *Let G be k-connected, $k\geq 2$, $T\subseteq V(G)$, $|T|\leq k+1=3$ for $k=2$ and $|T|\leq k+2$ for $k\geq 3$. Then G has a T-cycle if and only if G has no T-separator.*

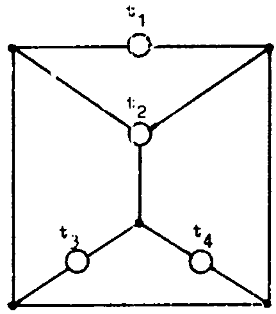

Fig. 1. $T=\{t_1, t_2, t_3, t_4\}$.

(In Theorem 1 of [4] the case $k=2$, $|T|=4$ should be excluded. The 2-connected graph in Fig. 1 has neither a T-cycle nor a T-separator for $T=\{t_1, \ldots, t_4\}$.)

2.3. Applying the characterization of the maximal number $q_G(X)$ of inner vertex disjoint chains between vertices of a given $X \subseteq V(G)$ [3, 6, 7] Theorem 2.2 can be reformulated as follows:

Theorem. *Let G be k-connected, $k \geq 2$, $T \subseteq V(G)$, $|T| \leq 3$ for $k=2$ and $|T| \leq k+2$ for $k \geq 3$. Then G has a T-cycle if and only if $q_G(T') \geq |T'|$ for any $T' \subseteq T$.*

2.4. Lemma. *Let G be k-connected, $k \geq 3$, $T \subseteq V(G)$, $|T| \geq k+1$, and let (X, Y) be a T-separator in G with $c_G(X, Y) = |T|-1$. Then $|X| \geq k-3+(3k-6)/(|T|-3)$.*

Proof. Since (X, Y) is a T-separator with $c_G(X, Y)=|T|-1$, $G-X-Y$ has $|T|$ components G_t, $t \in T$, with $t \in V(G_t)$. Since G is k-connected there exist $k \cdot |T|$ inner vertex disjoint chains from T to $X \cup G(Y)$. Let $G(Y_s)$ denote a component of $G(Y)$. Each vertex of ∂Y_s belongs to at most one G_t; therefore $k \cdot |T| \leq |X| \cdot |T| + \sum_s |\partial Y_s|$. Now the Lemma follows from

$$\sum_s |\partial Y_s| \leq 3 \sum_s \lfloor \tfrac{1}{2} |\partial Y_s| \rfloor = 3(|T|-|X|-1).$$

2.5. Remark. Assume the hypothesis of Lemma 2.4.

(1) Let $|T|=k+1$. Then by 2.4, $|X|=k$ so that $Y=\emptyset$. Thus by Theorem 2.2, if G has no T-cycle, then G has a T-separator of the form $(X, \emptyset)$ (see also [10]).

(2) Let $|T|=k+2$. Then by 2.4, $|X| \geq k$ for $k \geq 5$ and $|X| \geq k-1$ for $k=4$. In fact $|X| \geq k$ holds for $k \geq 4$ since the assumption $|X|=3$ for $k=4$ contradicts the fact that $G-X$ is connected.

(3) Let $|T|=k+3$. Then by 2.4, $|X| \geq k-6/k$ so that $|X| \geq k-2=1$ for $k=3$ and $|X| \geq k-1$ for $k \geq 4$. In fact $|X| \geq k$ holds for $k \geq 4$ since the assumption $|X|=k-1$ for $k \geq 4$ contradicts the fact that $G-X$ is connected.

The proof of Theorem 2.2 given in [4] is algorithmic. We gave an algorithm which takes a polynomial time to find either a T-cycle or a T-separator in a $(|T|-2)$-connected graph. Theorem 2.2 implies a number of corollaries. Some of them are listed in [4]. Some others are listed below.

3. The cycle-separator alternative fails for $|T| \geqslant \kappa(G)+3$

3.1. Here an example is given to show that Theorem 2.2 fails for $|T| = \kappa(G)+3$ (the example for $\kappa(G)=4$ given in [4] is wrong).

Consider pairs $[G, T]$ where $T \subseteq V(G)$. Let $[H_k, T_k]$ be obtained from $[Q, T]$ in Fig. 2 by adding two disjoint sets A, B of new vertices, $|A| = k-1$, $|B| = k$ and the edges connecting B with $A \cup V(Q)$. Here $T_k = T \cup A$.

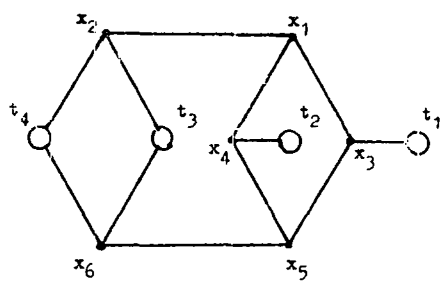

Fig. 2. $[Q, T]$ where $T = \{t_1, t_2, t_3, t_4\}$.

3.2. Proposition. *H_k is k-connected, $|T_k| = k+3$ and H_k has neither a T-cycle nor a T-separator.*

Proof. Clearly $|T_k| = k+3$ and H_k is k-connected.

(p1) Suppose that H_k has a T_k-cycle C. Then $C = C_1 \cup C_2$ where C_1, C_2 are chains with $C_1 \cap C_2 = \{b_1, b_2\} \subseteq B$, and $A \subseteq C_1$, $T \subseteq C_2$. Then the terminal edges of C_2 connect b_1, b_2 with $t_1, t_2 \in T$ while the inner edges of C_2 form a chain in Q between t_1, t_2 containing T. But surely no such chain exists in Q—contradiction.

(p2) Suppose that H_k has a T_k-separator (X, Y). Then obviously $B \subseteq X$. Let B' be a $(k-1)$-set of B. Then $(X-B', Y)$ is a T-separator of $H_k - B' - A$. On the other hand it is easy to check that for any subset $T' \subseteq T - A$, $H_k - B' - A$ has at least $|T'|$ inner vertex disjoint chains with end-points in T'. By the characterization of the maximal number of inner vertex disjoint chains between vertices of a given vertex subset of a graph [3, 6, 7] this implies that $H_k - B' - A$ has no T-separator—contradiction.

4. Another condition validating the cycle-seperator alternative

4.1. G is called *weakly separable k-connected* iff G is k-connected, and for any pair $(X, Y) \in P(G)$ (a) $c_G(X, Y) = k$ implies $\mathrm{comp}(G - X - Y) \leqslant k - 1$, and (b) $c_G(X, Y) = k + 1$ implies $\mathrm{comp}(G) \leqslant k + 1$. It is easy to prove the

Proposition. *Suppose that G is k-connected, $k \geqslant 3$, and $(X, Y) \in P(G)$. For $k \geqslant 4$, $Y \neq \emptyset$ implies* (a). *For $k \geqslant 3$, $c_{G-X}(Y) \geqslant 2$ implies* (b).

4.2. From Theorem 2.2 we have:

Corollary. *Let G be weakly separable k-connected and $k \geqslant 3$. Then any $k + 2$ or less vertices of G lie on a common cycle.*

4.3. Theorem. *Let G be weakly separable k-connected, $k \geqslant 3$, $T \subseteq V(G)$, and $|T| \leqslant k + 3$. Then G has no T-cycle iff $T = k + 3$ and G has a T-separator.*

For $k \geqslant 4$ the proof of this theorem follows the scheme of the proof of Theorem 2.2 in [4]. For $k = 3$ we derive the theorem from the following two auxiliary statements: under the hypothesis of the theorem, (1) if G has no T-separator, then T is covered by a union of at most three disjoint cycles of G each covering at least two members of T, and (2) if G has such a collection of cycles, then G has either a T-cycle or a T-separator.

In Sections 5 and 6 we apply Thoerem 4.3 to k-polytopal graphs and $C(m^*, n^*)$-graphs, respectively.

5. Any $k+2$ vertices of a k-polytopal graph lie on a common cycle

5.1. Definition. A graph G is called *k-polytopal*, $k \geqslant 2$, if G is isomorphic to the 1-skeleton of a k-dimensional convex polytope.

5.2. Lemma (M. Balinski [1]). *Let P be a k-dimensional convex polytope in $\mathbb{R}^k$, G be the 1-skeleton of P, H be a hyperplane of $\mathbb{R}^k$ cutting P and A be an open halfspace bounded by H. Let X be the set of vertices of G belonging to A. Then $G(X)$ is connected.*

Proof. Let H be described by $cx = a$ and A by $cx > a$. Let $V_{\max}$ be the set of vertices of G maximizing cx on P. Then obviously $V_{\max} \subseteq X$ and $G(V_{\max})$ is connected. It is well known that for any vertex $t \in V(G)$ there exists a path C of G from t to $v \in V_{\max}$ such that cx is being increased when moving along C from t to v. Therefore if $t \in X$, then $C \subseteq G(X)$ and so $G(X)$ is connected. □

5. Corollary. *Let G be a k-polytopal graph and $X \subseteq V(G)$. Then $G - X$ is connected for $|X| \leqslant k - 1$ and consists of at most 2 components for $|X| = k$.*

5.4. Lemma. *A k-polytopal graph, $k \geq 4$, is weakly separable k-connected.*

Proof. Let G be a k-polytopal graph. By Corollary 5.3, G is k-connected. By Proposition 4.1, it is sufficient to check 4.1(a) for $Y = \emptyset$ and 4.1(b) for $c_{G-X}(Y) = 0$ and 1. The validity of (a) for $Y = \emptyset$ and of (b) for $c_{G-X}(Y) = 1$ follows easily from Corollary 5.3. Let us now check (b) for $Y = \emptyset$ (whence $|X| = k+1$). Suppose that $G - X$ has at least $k+2$ components. By Lemma 5.2, X does not belong to a common hyperplane in $\mathbb{R}^k$. The conv X is a $(k+1)$-simplex. Let H_x, $x \in X$, denote the hyperplane containing $X - \{x\}$ and $\mathbb{R}_x$ denote the open halfspace of $\mathbb{R}^k$ bounded by H_x and not containing x. Then $\mathbb{R}^k = (\bigcup\{\mathbb{R}_x : x \in X\}) \cup \operatorname{conv} X$. Since $V(G) \cap \operatorname{conv} X = X$, each component of $G - X$ has a vertex in at least one R_x. Since the number of $\mathbb{R}_x$'s is $|X| = k+1$ while the number of components of $G - X$ is at least $k+2$, some $\mathbb{R}_y$ meets two components of $G - X$. Therefore $G - (X - \{y\})$ has at least two components in $\mathbb{R}_y$. This contradicts Lemma 5.2. $\square$

5.5. Using Theorem 5 from [4] (for $k = 3$), Corollary 4.2 and Lemma 5.4 we obtain the following result conjectured by G.T. Sallee [9]:

Corollary. *Any $k+2$ or less vertices in a k-polytopal graph lie on a common cycle.*

5.6. Theorem. *Let G be a k-polytopal graph, $k \geq 3$, $T \subseteq V(G)$ and $|T| \leq k+3$. Then*

(a) *G has no T-cycle if and only if $|T| = k+3$ and G has a T-separator; and*

(b) *a T-separator (if any) is of one of the following two forms (1) $(X, \emptyset)$ with $|X| = k+2$ or (2) (X, Y) with $|X| = k+1$, $G(Y)$ is connected and $|\partial_{G-X} Y| = 3$.*

The statement (a) follows from Theorem 5 in [4] for $k = 3$ and from Theorem 4.3 and Lemma 5.4 for $k \geq 4$. Now about (b). For $k = 3$ and $|T| = 6$ only T-separators of type (1) really occur (see Theorem 5 in [4]). Consider $k = 4$. By 2.5(3), $|X| \geq k - 1 = 3$. Suppose $|X| = 3$. Then X is seen to belong to at least five distinct 4-cuts $S_1, \ldots, S_5$ of G such that one of the two components of $G - S_i$, say C_i, is also a component of $G - X$, and $C_1, \ldots, C_5$ are all distinct. This is impossible in polytopal graphs. From the above reasoning and from 2.5(3) we have $|X| \geq k$ for $k \geq 4$. Further the assumption $|X| = k$ is easily seen to be impossible since $\operatorname{comp}(G - X) \leq 2$ by Corollary 5.3. Thus for $k \geq 4$, $|X| \geq k+1$.

This theorem generalizes Theorem 5 from [4] to higher dimensions and solves Problem 2 posed in [5].

6. Any $m+n$ vertices of a $C(m^+, n^-)$-graph lie on a common cycle, $2 \leq m \leq 5$

6.1. Definition (M.D. Plummer [7]). G is $C(m^+, n^-)$ if $|V(G)| \geq m+n$ and for any n-subset Z of $V(G)$ any m vertices of $G - Z$ lie on a common cycle.

6.2. Lemma. *Suppose that G is $C(m^+, n^-)$, $m \geqslant 2$. Then*

(1) *G is $(n+2)$-connected.*

(2) *If $m \geqslant i \in \{3, 4\}$, then for $(X, Y) \in P(G)$ such that $|X| \geqslant n$ and $c_G(X, Y) = n+i-1$ we have $\mathrm{comp}(G-X-Y) \leqslant i-1$ (so that for $m \geqslant 4$ G is weakly separable $(n+2)$-connected).*

Proof. Since $m \geqslant 2$, G is $(n+2)$-connected. Consider $(X, Y) \in P(G)$ with $|X| \geqslant n$. Let Z be an n-subset of X. Suppose $m \geqslant i \in \{3, 4\}$ and $c_G(X, Y) = n+i-1$. Assume that $G-X-Y$ has i components $A_1, \ldots, A_i$. Since $c_{G-Z}(X-Z, Y) = i-1$, we have: $(X-Z, Y)$ is a T-separator in $G-Z$ for any i-subset $T \subseteq V(G-Z)$ meeting $A_1, \ldots, A_i$. Therefore T does not lie on a common cycle in $G-Z$ and so G is not $C(i^+, n^-)$—contradiction. ☐

6.3. Theorem. *Suppose that G is $C(m^+, n^-)$, $n \geqslant 1$, $2 \leqslant m \leqslant 5$. Then any $m+n$ vertices of G lie on a common cycle.*[1]

Proof. For $m = 2$ the theorem follows from the Dirac theorem [2]. For $m = 3, 4$ the theorem follows immediately from Theorem 2.2, Remark 2.5 and Lemma 6.2. Now let $m = 5$. Suppose G has no T-cycle for some subset T of $n+5$ vertices. By Lemma 6.2, G is weakly separable $(n+2)$-connected. Therefore G has a T-separator (X, Y) by Theorem 4.3, with $|X| \geqslant n+3$, by 2.5(3). Let Z be an n-subset of X and T' be a 5-subset of T. Then $(X-Z, Y)$ is a T'-separator in $G-Z$ and so G is not $C(5^+, n^-)$—contradiction. ☐

6.4. Let $M(n)$ denote the set of integers m such that $C(m^+, n^-)$ implies $C((m+n)^+, 0^-)$. In [5] the problem is posed to describe $M(n)$. Theorem 6.3 asserts that $\{2, 3, 4, 5\} \subseteq M(n)$. The Petersen graph shows that $9 \notin M(1)$. The *question* remains whether or not $i \in M(1)$ for $i \in \{6, 7, 8\}$.

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